On free general relativistic initial data on the light cone

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October 12, 2010

Abstract

We provide a simple explicit parameterization of free general relativistic data on the light cone.

Contents

1 Introduction

In a series of recent papers [1–4], solutions of the vacuum Einstein equations defined to the future of a light cone, say C_O, issued from a point O, have been characterized in terms of data on a light cone. Part of those data is provided by a symmetric degenerate tensor on C_O, and the approach there requires this degenerate tensor to be induced on C_O by *some* smooth Lorentzian metric $C = C_{\mu\nu}dx^\mu dx^\nu$. The question then arises, how to usefully describe the induced tensors having this property. Now, tensor fields on $(0,R)\times S^2$ with vanishing r–components, where r parameterizes $(0,R)$, can always be written in the form (see, e.g., [7, Appendix E])

$$r^2\big[(1+\gamma)\mathring{s}_{AB} + 2\alpha_{||AB} - \mathring{s}_{AB}\mathring{s}^{CD}\alpha_{||CD} + \mathring{\epsilon}_A{}^C\beta_{||CB} + \mathring{\epsilon}_B{}^C\beta_{||CA}\big]dx^A dx^B\ , \quad (1.1)$$

where $\mathring{s} \equiv \mathring{s}_{AB}dx^A dx^B$ is the round unit metric on S^2, and $||$ denotes covariant differentiation on $(S^2,\mathring{s})$. Further, $\mathring{s}^{AB}$ is the inverse metric to $\mathring{s}_{AB}$, $\mathring{\epsilon}^A{}_B := \mathring{s}^{AC}\mathring{\epsilon}_{CB}$, and $\mathring{\epsilon}_{AB}$ is the alternating tensor on $(S^2,\mathring{s})$. This shifts the extendibility question to that of the properties of the functions α, β and γ. The aim of this note is to prove the following (see Section 2.1 for terminology and Section 2.2 for the proof):

Theorem 1.1 *A tensor field on* $(0,R)\times S^2$ *of the form* (1.1) *is the restriction of a smooth metric in normal coordinates to its light cone if and only if the functions* α, β *and* γ *are* C_O*–smooth, except possibly for the* $\ell = 0$ *and* $\ell = 1$ *spherical harmonics of* α *and* β *which give zero contribution to* (1.1)*.*

Consider the vacuum general relativistic characteristic constraint equation in the affinely parameterized gauge (see, e.g., [3]):

$$\partial_1\tau + \frac{\tau^2}{n-1} + |\sigma|^2 = 0\ , \quad (1.2)$$

where τ is the divergence of C_O and σ its shear. Given α and β, Equation (1.2) can be viewed as a non-linear ODE for γ, and thus the functions α and β can be thought of as representing unconstrained degrees of freedom of the gravitational field.

Theorem 1.1 invokes normal coordinates for the metric C, and its proof requires a useful description of the components of a metric tensor in normal coordinates. This is provided by the following result, proved in Section 2.1, which has some interest of its own:

Theorem 1.2 *The coordinates w^μ are normal for a metric $C_{\mu\nu}$ if and only if there exists a tensor field $\Omega_{\alpha\beta\gamma\delta}$ satisfying*

$$\Omega_{\alpha\beta\gamma\delta} = \Omega_{\gamma\delta\alpha\beta} = -\Omega_{\beta\alpha\gamma\delta} \tag{1.3}$$

such that

$$\underline{g_{\alpha\gamma}} = \underline{\eta_{\alpha\gamma}} + \underline{\Omega_{\alpha\beta\gamma\delta}} w^\beta w^\delta \ , \tag{1.4}$$

where underlined tensor components denote coordinate components in the coordinate system w^μ, and where η is the Minkowski metric.

Remark 1.3 While we are mainly interested in Lorentzian metrics, we note that Theorem 1.2 has a direct counterpart in all signatures.

The main issue of our work is the understanding of the behaviour of the objects at hand near the vertex of the cone. Many of the considerations below are valid only within the domain of definition of normal coordinates centered at the vertex of the light cone, which is sufficient for the purpose.

2 Tensors and the light cone

Consider a smooth metric C in normal coordinates w^μ. As already pointed out, we write $\underline{C_{\beta\gamma}}$ for the coordinate components of the metric tensor in this coordinate system. We reserve the notation $C_{\mu\nu}$ for the components of C in the coordinate system $(x^0 \equiv u, x^1 \equiv r, x^A)$, defined as

$$w^0 = x^1 - x^0, \qquad w^i = x^1 \Theta^i(x^A) \qquad \text{with} \quad \sum_{i=1}^n \left[\Theta^i(x^A)\right]^2 = 1 \ . \tag{2.1}$$

Thus

$$\partial_u = -\partial_{w^0} \ , \quad \partial_r = \partial_{w^0} + \frac{w^i}{r}\partial_{w^i} \ ,$$

and

$$\eta = -(dx^0)^2 + 2dx^0 dx^1 + r^2 \mathring{s}_{AB} dx^A dx^B \ , \quad \eta^\sharp = \partial_r^2 + 2\partial_u\partial_r + r^{-2}\mathring{s}^{AB}\partial_A\partial_B \ .$$

The explicit form of the transformation formulae for a symmetric tensor $T_{\mu\nu}$ reads

$$T_{00} \equiv \underline{T_{00}}, \quad T_{01} \equiv -\underline{T_{00}} - \underline{T_{0i}}\Theta^i, \quad T_{0A} \equiv -\underline{T_{0i}} r \frac{\partial \Theta^i}{\partial x^A} \ , \tag{2.2}$$

$$T_{11} \equiv \underline{T_{00}} + 2\underline{T_{0i}}\Theta^i + \underline{T_{ij}}\Theta^i\Theta^j , \quad T_{1A} \equiv \underline{T_{0i}} r \frac{\partial \Theta^i}{\partial x^A} + \underline{T_{ji}} r \Theta^j \frac{\partial \Theta^i}{\partial x^A} , \tag{2.3}$$

$$T_{AB} \equiv \underline{T_{ij}} r^2 \frac{\partial \Theta^i}{\partial x^A}\frac{\partial \Theta^j}{\partial x^B} . \tag{2.4}$$

Conversely, $\underline{T_{\lambda\mu}} = \frac{\partial x^\alpha}{\partial w^\lambda}\frac{\partial x^\beta}{\partial w^\mu} T_{\alpha\beta}$ gives

$$\underline{T_{00}} \equiv T_{00}, \quad \underline{T_{0i}} \equiv -(T_{00}+T_{01})\Theta^i - T_{0A}\frac{\partial x^A}{\partial w^i} , \tag{2.5}$$

$$\underline{T_{ij}} = (T_{00}+2T_{01}+T_{11})\Theta^i\Theta^j + (T_{0A}+T_{1A})\left(\Theta^i\frac{\partial x^A}{\partial w^j} + \Theta^j \frac{\partial x^A}{\partial w^i}\right) + T_{AB}\frac{\partial x^A}{\partial w^i}\frac{\partial x^B}{\partial w^j} . \tag{2.6}$$

An overline over a function f denotes restriction of the function to the light cone $C_O = \{w^0 = |\vec{w}|\}$: if we parameterize the cone by $\vec{w} \equiv (w^i)$, we have

$$\overline{f}(\vec{w}) := f(w^0 = |\vec{w}|, \vec{w}) ,$$

where $|\vec{w}|^2 := \sum_i (w^i)^2$.

Note that the domain of definition of normal coordinates for a general metric is rarely global, and that our considerations apply only within this domain.

Since C_O can be coordinatised by $\vec{w}$, functions on C_O can be identified with functions of $\vec{w}$. A function φ on C_O will be said to belong to $C^k(C_O)$ if φ can be written as $\hat{\varphi} + r\check{\varphi}$, where $\hat{\varphi}$ and $\check{\varphi}$ are C^k functions of $\vec{w}$. A function on C_O will be called *C_O–smooth* if it can be written as $\hat{\varphi} + r\check{\varphi}$, where $\hat{\varphi}$ and $\check{\varphi}$ are smooth functions of $\vec{w}$. A similar definition is used for real-analytic functions. It is not too difficult to show that a function φ is C_O–smooth if and only if there exists a smooth function φ on space-time such that $\varphi = \overline{f}$. In other words:

Proposition 2.1 *A function φ defined on*

$$C_O := \left\{ w^\mu \in \mathbf{R}^{n+1} : \ w^0 = \sqrt{\sum_i (w^i)^2} \right\}$$

can be extended to a C^k, respectively smooth, respectively analytic, function on $\mathbf{R}^{n+1}$ if and only if φ is $C^k(C_O)$, respectively C_O–smooth, respectively C_O–analytic.

The proof of Proposition 2.1 for real-analytic functions can be found in [2]; the remaining cases are covered in Appendix A.

2.1 Normal coordinates

Recall that (local) coordinates w^μ are normal for the metric C if and only if it holds that [10]

$$\underline{C_{\mu\nu}}w^\mu = \underline{\eta_{\mu\nu}}w^\mu \ . \tag{2.7}$$

For completeness, and because of restricted accessibility of [10], we give a proof of this in Appendix B.

It follows from (2.2) and (2.7) that

$$\begin{aligned}
\overline{C_{11}} &= \frac{1}{r^2}\overline{\underline{C_{\mu\nu}w^\mu w^\nu}} = \frac{1}{r^2}\overline{\underline{\eta_{\mu\nu}}w^\mu w^\nu} = 0 \ , && (2.8)\\
\overline{C_{01}} &= -\frac{1}{r}\overline{\underline{C_{0\nu}}w^\nu} = -\frac{1}{r}\overline{\underline{\eta_{0\nu}}w^\nu} = 1 \ , && (2.9)\\
\overline{C_{1A}} &= \overline{\underline{C_{i\mu}}w^\mu\frac{\partial\Theta^i}{\partial x^A}} \\
&= \overline{\underline{\eta_{i\mu}}w^\mu\frac{\partial\Theta^i}{\partial x^A}} = r\sum_i\Theta^i\frac{\partial\Theta^i}{\partial x^A} = \frac{1}{2}\sum_i r\frac{\partial(\Theta^i\Theta^i)}{\partial x^A} = 0 && (2.10)
\end{aligned}$$

(note that the only information, that does not immediately follow from the fact that C_O is the future light cone for the metric C, is provided by(2.9); the remaining equations can serve as consistency checks).

We set

$$h_{\mu\nu} := C_{\mu\nu} - \eta_{\mu\nu} \ ,$$

and we will lower and raise all indices with the metric η. Hence the coordinates w^α are normal for $C = C_{\mu\nu}dw^\mu dw^\nu$ if and only if

$$\underline{h_{\mu\nu}}w^\mu = 0 \ . \tag{2.11}$$

Note that, from (2.8)-(2.10),

$$\overline{h_{1\mu}} = 0 \quad \Longleftrightarrow \quad \overline{h^{0\mu}} := \eta^{0\alpha}\eta^{\mu\beta}h_{\alpha\beta} = 0 \ . \tag{2.12}$$

The question arises, how to describe exhaustively, and in a useful way, the set of tensors satisfying (2.11). One obvious way of doing this is to use a projection operator: indeed, for any smooth symmetric tensor $\phi_{\mu\nu}$, the tensor field

$$P_\alpha{}^\mu P_\beta{}^\nu(\eta_{\rho\sigma}w^\rho w^\sigma)^2\phi_{\mu\nu} \ , \text{ where } P_\alpha{}^\beta = \delta_\alpha^\beta - \frac{\eta_{\alpha\mu}w^\mu w^\beta}{\eta_{\rho\sigma}w^\rho w^\sigma}$$

is a smooth tensor field satisfying (2.11). This leads to a restricted class of tensors because of the multiplicative factor $(\eta_{\rho\sigma}w^\rho w^\sigma)^2$ above (in particular the resulting tensor induced on the light cone has vanishing AB components), and it is not clear how to guarantee smoothness of the final result without the multiplicative factor. Variations on the above using a space projector $\delta^i_j - r^{-2}x^i x^j$ lead to similar difficulties.

Note, however, that solutions of (2.11) can be constructed as follows: let $\Omega_{\alpha\beta\gamma\delta}$ be any smooth tensor field satisfying (1.3). Then the tensor field

$$\underline{h_{\alpha\gamma}} = \underline{\Omega_{\alpha\beta\gamma\delta}} w^\beta w^\delta \tag{2.13}$$

is symmetric, and satisfies (2.11). Theorem 1.2 follows now immediately from:

Proposition 2.2 *A tensor field $h_{\mu\nu}$ satisfies* (2.11) *if and only if there exists a tensor field $\Omega_{\alpha\beta\gamma\delta}$ satisfying* (1.3) *such that* (2.13) *holds.*

Proof. We work in a given smooth coordinate system x^μ. The sufficiency has already been established. To show necessity recall, first, that any smooth tensor field satisfying

$$A_\mu x^\mu = 0 \tag{2.14}$$

can be represented as

$$A_\mu = \Omega_{\mu\nu}x^\nu \ , \ \text{with } \Omega_{\mu\nu} = -\Omega_{\nu\mu} \ .$$

To see this, note first that differentiation of (2.14) shows that $A_\mu(0) = 0$; then

$$\begin{aligned} A_\mu(x^\sigma) &= \int_0^1 \frac{d}{ds}\left[sA_\mu(sx^\sigma)\right] ds = \int_0^1 \left[A_\mu(sx^\sigma) + sx^\nu\partial_\nu A_\mu(sx^\sigma)\right] ds \\ &= x^\nu \underbrace{\int_0^1 s(\partial_\nu A_\mu - \partial_\mu A_\nu)(sx^\sigma)ds}_{=:\Omega_{\mu\nu}} , \end{aligned}$$

where we have used

$$\partial_\mu(x^\nu A_\nu) = 0 \quad \Longrightarrow \quad A_\mu(sx^\sigma) = -sx^\nu\partial_\mu A_\nu(sx^\sigma) \ .$$

Applying this to $h_{\mu\nu}$ at fixed ν we find that there exists a field $\Omega_{\alpha\beta\nu}$, anti-symmetric in α and β, such that

$$h_{\mu\nu}(x^\rho) = \Omega_{\mu\alpha\nu}(x^\rho)x^\alpha \ .$$

Applying the construction again to the last equation at fixed μ and α we conclude that

$$\Omega_{\mu\alpha\nu}(x^\rho) = \Omega_{\mu\alpha\nu\beta}(x^\rho)x^\beta$$

for some field $\Omega_{\alpha\beta\gamma\delta}$, anti-symmetric in γ and δ. This is of the desired form, but the pair-interchange symmetry is not completely clear. However, the above prescription gives

$$\Omega_{\mu\sigma\nu\lambda}(x^\rho) = \int_0^1 s^2 \int_0^1 t\left(\partial_\lambda\partial_\sigma h_{\mu\nu} - \partial_\lambda\partial_\mu h_{\sigma\nu} + \partial_\nu\partial_\mu h_{\sigma\lambda} - \partial_\nu\partial_\sigma h_{\mu\lambda}\right)(stx^\rho)\, dt\, ds \;, \tag{2.15}$$

which makes manifest all the symmetries claimed. This equation defines the components of the tensor field $\Omega_{\mu\sigma\nu\lambda}(x^\rho)$ in the coordinate system x^μ. $\square$

One should bear in mind that $\Omega_{\alpha\beta\gamma\delta}$ is not uniquely defined by (2.13). However, (2.15) can be used as a canonical choice, if needed.

It would be of interest to provide an answer to the corresponding question for tensor fields satisfying (2.11) on the light cone only:

$$\overline{\underline{h_{\mu\nu}}w^\mu} = 0 \;. \tag{2.16}$$

We return to this question in Section 3, where some partial results are given, but we have not attempted an exhaustive study. In any case, on the light cone (2.11) gives the following:

$$\overline{\underline{h_{00}}} = \overline{\underline{\Omega_{0i0j}}}w^iw^j \;, \tag{2.17}$$

$$\overline{\underline{h_{0i}}} = \big(-\overline{\underline{\Omega_{0j0i}}}r + \overline{\underline{\Omega_{0jik}}}w^k\big)w^j \;, \tag{2.18}$$

$$\overline{\underline{h_{ij}}} = \overline{\underline{\Omega_{i0j0}}}r^2 + \big[-\overline{\underline{\Omega_{0ijk}}}r - \overline{\underline{\Omega_{0jik}}}r + \overline{\underline{\Omega_{ikj\ell}}}w^\ell\big]w^k \;. \tag{2.19}$$

In coordinates adapted to the light cone (2.17)-(2.19) translate to

$$\overline{h_{00}} = \overline{\underline{\Omega_{0i0j}}}w^iw^j \;, \tag{2.20}$$

$$\overline{h_{\mu1}} = 0 \;, \tag{2.21}$$

$$\overline{h_{0A}} = r\big(\overline{\underline{\Omega_{0j0i}}}r - \overline{\underline{\Omega_{0jik}}}w^k\big)w^j\frac{\partial\Theta^i}{\partial x^A} \;, \tag{2.22}$$

$$\overline{h_{AB}} = r^2\bigg(\overline{\underline{\Omega_{i0j0}}}r^2 + \big(-\overline{\underline{\Omega_{0ijk}}}r - \overline{\underline{\Omega_{0jik}}}r + \overline{\underline{\Omega_{ikj\ell}}}w^\ell\big)w^k\bigg)\frac{\partial\Theta^i}{\partial x^A}\frac{\partial\Theta^j}{\partial x^B} \;. \tag{2.23}$$

In particular $\overline{h_{0A}}$ factors out through r and is $O(r^3)$, while $\overline{h_{AB}}$ factors out through r^2 and is $O(r^4)$.

For further use we note

$$\begin{aligned}
&\overline{h_{AB}}\frac{\partial x^A}{\partial w^p}\frac{\partial x^B}{\partial w^q}\\
&= \underline{\overline{\Omega_{i0j0}}}(r\delta^i_p - w^i\Theta^p)(r\delta^j_q - w^j\Theta^q)\\
&\quad +\big[-\underline{\overline{\Omega_{0iqk}}}(r\delta^i_p - w^i\Theta^p) - \underline{\overline{\Omega_{0jpk}}}(r\delta^j_q - w^j\Theta^q) + \underline{\overline{\Omega_{pkq\ell}}}w^\ell\big]w^k\\
&= \underline{\overline{\Omega_{i0j0}}}w^iw^j\Theta^p\Theta^q + w^kw^i\big(\underline{\overline{\Omega_{0iqk}}}\Theta^p + \underline{\overline{\Omega_{0ipk}}}\Theta^q\big) \qquad (2.24)\\
&\quad +r^2\underline{\overline{\Omega_{p0q0}}} - \underline{\overline{\Omega_{i0q0}}}w^iw^p - \underline{\overline{\Omega_{p0i0}}}w^iw^q - \big(r\underline{\overline{\Omega_{0pqk}}} + r\underline{\overline{\Omega_{0qpk}}} - \underline{\overline{\Omega_{pkq\ell}}}w^\ell\big)w^k\ .
\end{aligned}$$

This equation has been derived under the assumption that the coordinates w^μ are normal; however, $\overline{h_{AB}}dx^Adx^B$ is intrinsic to the light cone, and hence this equation provides the most general form of a tensor field $\overline{h_{AB}}dx^Adx^B$ arising from some smooth metric $C_{\mu\nu}$ in coordinates which coincide with the normal ones on the light cone C_O.

Note that given the specific structure of the terms containing Θ^i above, it is clear how to extract $\underline{\overline{\Omega_{0i0j}}}$ and $\underline{\overline{\Omega_{0ijk}}}$ from h_{AB}.

We shall say that a tensor field h is C_O–smooth if there exists a coordinate system w^μ in which the components of h are C_O–smooth. We conclude that (keeping in mind the local character of normal coordinates):

Proposition 2.3 *A tensor field $\varphi_{AB}dx^Adx^B$ on C_O arises from the restriction to the light cone of a metric in normal coordinates if and only if there exist C_O–smooth tensor fields A_{ij}, symmetric in its indices, A_{ijk}, anti-symmetric in the last two indices, and A_{ijkl}, satisfying $A_{ijkl} = A_{klij} = -A_{jikl}$, such that*

$$\begin{aligned}
&\big(\varphi_{AB} - r^2\mathring{s}_{AB}\big)\frac{\partial x^A}{\partial w^p}\frac{\partial x^B}{\partial w^q}\\
&= \underline{A_{ij}}w^iw^j\Theta^p\Theta^q + w^k\big(\underline{A_{iqk}}w^i\Theta^p + \underline{A_{jpk}}w^j\Theta^q\big)\\
&\quad +r^2\underline{A_{pq}} - \underline{A_{iq}}w^iw^p - \underline{A_{pi}}w^iw^q - \big(r\underline{A_{pqk}} + r\underline{A_{qpk}} - \underline{A_{pkq\ell}}w^\ell\big)w^k\ . \qquad (2.25)
\end{aligned}$$

Proof. The necessity is clear from (2.24). To show sufficiency, suppose that a tensor field satisfying (2.25) is given. Let $\Omega_{\mu\nu\rho\sigma}$ be any smooth tensor field satisfying $\Omega_{\mu\nu\rho\sigma} = -\Omega_{\nu\mu\rho\sigma} = \Omega_{\rho\sigma\mu\nu}$ such that

$$\underline{\overline{\Omega_{0i0j}}} = \underline{A_{ij}}\ ,\quad \underline{\overline{\Omega_{0ijk}}} = \underline{A_{ijk}}\ ,\quad \underline{\overline{\Omega_{ijkl}}} = \underline{A_{ijkl}}\,;$$

existence of $\Omega_{\alpha\beta\gamma\delta}$ follows from Proposition 2.1. Then φ_{AB} is the restriction to the light cone of the smooth tensor field $\underline{\eta_{\mu\nu}} + \underline{\Omega_{\mu\rho\nu\sigma}} w^\rho w^\sigma$ for which the coordinates w^μ are normal. □

Recall [3] (compare [8, 9]) that solutions of the Cauchy problem for the vacuum Einstein equations with initial data on an affinely-parameterized light cone are uniquely determined by the conformal class of $\overline{C_{AB}} dx^A dx^B$. The remaining components of $C_{\mu\nu}$ are thus irrelevant for that purpose, and for the sake of computations it is convenient to choose them as simple as possible. It is therefore of interest to enquire whether any $\overline{C_{AB}}$ can be realized by a smooth metric satisfying

$$\underline{\overline{C_{00}}} = -1 \ , \quad \underline{\overline{C_{0i}}} = 0 \ , \quad \underline{\overline{C_{ij}}} w^j = w^i \ . \tag{2.26}$$

Our equations above show that this is only possible for C_{AB}'s which, in coordinates which coincide with the normal ones on C_O, are of the form

$$\overline{h_{AB}} \frac{\partial x^A}{\partial w^p} \frac{\partial x^B}{\partial w^q} \quad = \quad \underline{\overline{\Omega_{pkq\ell}}} w^\ell w^k \ .$$

Equivalently, all the functions $\overline{h_{AB}} \frac{\partial x^A}{\partial w^p} \frac{\partial x^B}{\partial w^q}$ are C_O–smooth.

We finish this section by the following curious observation, which shows that normal coordinates can be induced from one-dimension-up:

Proposition 2.4 *The coordinates $w^i|_{w^0=0}$ are normal for the metric*

$$g_{ij}|_{w^0=0} dw^i dw^j \ .$$

Proof. From $h_{\mu\nu} w^\mu = 0$ one finds $h_{ij}|_{w^0=0} w^i = 0$, and the result follows from the Riemannian counterpart of the equivalence (2.11). □

2.2 Scalar potentials for the metric in dimension $3+1$

So far we have been using general space dimension n. For $n = 3$, using a standard decomposition (cf., e.g., [7]) of symmetric tensors on $S^{n-1} = S^2$ we can write

$$\overline{C_{AB}} = r^2 \big[(1+\gamma) \mathring{s}_{AB} + 2\alpha_{||AB} - \mathring{s}_{AB} \mathring{s}^{CD} \alpha_{||CD} + \mathring{\epsilon}_A{}^C \beta_{||CB} + \mathring{\epsilon}_B{}^C \beta_{||CA} \big] , \tag{2.27}$$

We wish to find necessary and sufficient conditions on the functions α, β and γ so that $\overline{C_{AB}}$ arises from a smooth metric on space-time.

For reasons that will become apparent shortly, we want to calculate

$$\eta^{\alpha\mu}\eta^{\beta\nu}\mathring{\nabla}_\alpha\mathring{\nabla}_\beta C_{\mu\nu} \quad\text{and}\quad \eta^{\sigma\rho}T_\alpha w_\beta \epsilon^{\alpha\beta\gamma\delta}\mathring{\nabla}_\rho\mathring{\nabla}_\gamma C_{\delta\sigma}\ ,$$

where $\mathring{\nabla}$ is the covariant derivative of the metric η, while

$$\underline{T_\alpha} := \underline{\eta_{\alpha 0}} = -\delta^0_\alpha\ , \quad \underline{w_\alpha} = \underline{\eta_{\alpha\beta}w^\beta}\ .$$

The calculation of $\eta^{\alpha\mu}\eta^{\beta\nu}\mathring{\nabla}_\alpha\mathring{\nabla}_\beta C_{\mu\nu}$ can, and will, be done without assuming $n=3$; we will use the symbol $\mathring{s}$ to denote the unit round metric on S^{n-1}. Writing $(x^a)=(x^0,x^1)$, from

$$\mathring{\Gamma}^A_{rB} = \frac{1}{r}\delta^A_B\ , \quad \mathring{\Gamma}^u_{AB} = -\frac{1}{r}\eta_{AB} = \mathring{\Gamma}^r_{AB}\ ,$$

we find

$$\begin{aligned}
\mathring{\nabla}_\mu h^{\mu\nu} &:= \eta^{\mu\sigma}\eta^{\nu\beta}\mathring{\nabla}_\sigma h_{\alpha\beta}\\
&= \partial_A h^{A\nu} + \partial_a h^{a\nu} + 2h^{rB}\mathring{\Gamma}^\nu_{Br} + \frac{n-1}{r}h^{\nu r} + h^{\nu A}\mathring{\Gamma}^B_{AB} + h^{AB}\mathring{\Gamma}^\nu_{AB}\ .
\end{aligned}$$

Hence

$$\mathring{\nabla}_\mu h^{\mu b} = h^{Ab}{}_{||A} + \partial_a h^{ab} + \frac{n-1}{r}h^{br} - \frac{1}{r}h_{AB}\eta^{AB}\ , \tag{2.28}$$

where $||$ denotes covariant differentiation on $(S^{n-1},\mathring{s})$. Further, using $\mathring{\nabla}_\mu X^\mu = |\det\eta|^{-1/2}\partial_\mu(|\det\eta|^{1/2}X^\mu)$,

$$\begin{aligned}
\eta^{\alpha\mu}\eta^{\beta\nu}\mathring{\nabla}_\alpha\mathring{\nabla}_\beta h_{\mu\nu} = {}& h^{AB}{}_{||AB} + h^{ab}{}_{,ab} + 2\partial_a h^{aA}{}_{||A} + \frac{n+3}{r}h^{rA}{}_{||A} + \frac{n+1}{r}\partial_a h^{ra}\\
&-\frac{1}{r}(\partial_u H + \partial_r H) - \frac{1}{r^2}H + \frac{n-1}{r^2}h^{rr}\ ,
\end{aligned} \tag{2.29}$$

and

$$H := \eta^{AB}h_{AB}\ , \quad\text{hence}\quad \overline{H} = \overline{\eta^{\mu\nu}h_{\mu\nu}}\ .$$

To analyze the right-hand side of (2.29) the following formulae are useful:

$$h^{rr} = h_{uu} + 2h_{ur} + h_{rr} = \underline{h_{ij}}\Theta^i\Theta^j\ , \tag{2.30}$$

$$h^{ur} = h_{ur} + h_{rr} = \underline{h_{0i}}\Theta^i + \underline{h_{ij}}\Theta^i\Theta^j\ , \tag{2.31}$$

$$h^{uA} = h_{rA} = \underline{h_{0i}}r\frac{\partial\Theta^i}{\partial x^A} + \underline{h_{ji}}w^j\frac{\partial\Theta^i}{\partial x^A}\ , \tag{2.32}$$

$$h^{uu} = h_{rr} = \underline{h_{00}} + 2\underline{h_{0i}}\Theta^i + \underline{h_{ij}}\Theta^i\Theta^j\ , \tag{2.33}$$

$$H \equiv \eta^{AB}h_{AB} = \eta^{\mu\nu}h_{\mu\nu} + \underline{h_{00}} - \underline{h_{ij}}\Theta^i\Theta^j\ . \tag{2.34}$$

Functions of the form $r^{-2}(\mu+r\nu)$, where μ and ν are restrictions to the light cone of smooth functions on space-time, will be called *mildly singular*. In what follows one should keep in mind that any function φ can be written as $r^2\varphi/r^2$, and is thus mildly singular if φ is C_O–smooth. In particular, all h_{ab}'s and h^{ab}'s are mildly singular if the metric C is smooth.

Denoting by "m.s." the sum of all mildly singular terms that might occur, one finds

$$
\begin{aligned}
\partial_a\partial_b h^{ab} &= \Theta^i\Theta^j\Theta^k\Theta^\ell\partial_{w^k}\partial_{w^\ell}\underline{h_{ij}} + \text{m.s.}\ , \\
2\partial_a h^{aB}{}_{||B} &= -2\Theta^i\Theta^j\Theta^k\Theta^\ell\partial_{w^i}\partial_{w^j}\underline{h_{ij}} - \frac{2n}{r}\Theta^i\Theta^j\Theta^k\partial_{w^k}\underline{h_{ij}} \\
&\quad + \frac{2n}{r^2}\Theta^i\Theta^j\underline{h_{ij}} + \text{m.s.}\ , \\
\frac{n+3}{r}h^{rB}{}_{||B} &= -\frac{n+3}{r}\Theta^i\Theta^j\Theta^k\partial_{w^k}\underline{h_{ij}} - \frac{n(n+3)}{r^2}\Theta^i\Theta^j\underline{h_{ij}} + \text{m.s.}\ , \\
\frac{n+1}{r}\partial_a h^{ar} &= \frac{n+1}{r}\Theta^i\Theta^j\Theta^k\partial_{w^k}\underline{h_{ij}} + \text{m.s.}\ , \\
-\frac{1}{r}(\partial_r H + \partial_u H) &= \frac{1}{r}\Theta^i\Theta^j\Theta^k\partial_{w^k}\underline{h_{ij}} + \text{m.s.}\ , \\
\frac{n-1}{r^2}h^{rr} - \frac{1}{r^2}H &= \frac{n}{r^2}\Theta^i\Theta^j\underline{h_{ij}} + \text{m.s.}\ .
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
h^{AB}{}_{||AB} &= -\partial_{w^i}\partial_{w^j}\underline{h_{k\ell}}\Theta^i\Theta^j\Theta^k\Theta^\ell - \frac{(2n+1)}{r}\partial_{w^i}\underline{h_{jk}}\Theta^i\Theta^j\Theta^k \\
&\quad -\frac{n^2}{r^2}\underline{h_{ij}}\Theta^i\Theta^j + \text{m.s.}\ .
\end{aligned} \tag{2.35}
$$

We emphasize that this formula is independent of the "gauge condition" $\underline{h_{\mu\nu}}w^\mu = 0$.

We now assume that the space dimension n equals three. From (2.27) we find

$$
\gamma = \frac{\overline{H}}{2} = \frac{1}{2}\overline{\left(\eta^{\mu\nu}h_{\mu\nu} + \underline{h_{00}} - \underline{h_{ij}}\Theta^i\Theta^j\right)}\ , \tag{2.36}
$$

which is mildly singular. Let χ_{AB} denote the $\mathring{s}$–trace-free part of h_{AB}, then

$$
h^{AB}{}_{||AB} = \chi^{AB}{}_{||AB} + \frac{1}{r^2}\mathring{\Delta}\gamma\ ,
$$

where $\mathring{\Delta}$ is the Laplace-Beltrami operator of $\mathring{s}$. With some work, using $\mathring{\Delta}\Theta^i = -2\Theta^i$, we find

$$\begin{aligned}\frac{1}{r^2}\mathring{\Delta}\gamma &= -\frac{1}{2}\partial_{w^i}\partial_{w^j}\underline{h_{k\ell}}\Theta^i\Theta^j\Theta^k\Theta^\ell - \frac{3}{r}\partial_{w^i}\underline{h_{jk}}\Theta^i\Theta^j\Theta^k \\ &\quad -\frac{3}{r^2}\underline{h_{ij}}\Theta^i\Theta^j + \text{m.s.}\ ,\end{aligned} \tag{2.37}$$

which shows that $\chi^{AB}{}_{||AB}$ is again of the general form (2.35):

$$\begin{aligned}\chi^{AB}{}_{||AB} &= -\frac{1}{2}\partial_{w^i}\partial_{w^j}\underline{h_{k\ell}}\Theta^i\Theta^j\Theta^k\Theta^\ell - \frac{4}{r}\partial_{w^i}\underline{h_{jk}}\Theta^i\Theta^j\Theta^k \\ &\quad -\frac{6}{r^2}\underline{h_{ij}}\Theta^i\Theta^j + \text{m.s.}\ .\end{aligned} \tag{2.38}$$

It turns out that things improve when the normal coordinates condition is invoked. For then we have

$$\underline{h_{ij}}w^i = -\underline{h_{0j}}w^0\ , \tag{2.39}$$

$$\underline{h_{0j}}w^i = -\underline{h_{00}}w^0\ , \tag{2.40}$$

$$\underline{h_{ij}}w^iw^j = \underline{h_{00}}(w^0)^2\ , \tag{2.41}$$

$$w^kw^iw^j\partial_k\underline{h_{ij}} = \big(-2\underline{h_{00}} + w^k\partial_k\underline{h_{00}}\big)(w^0)^2\ , \tag{2.42}$$

$$w^\ell w^kw^iw^j\partial_\ell\partial_k\underline{h_{ij}} = \left[w^\ell\partial_\ell\big(-2\underline{h_{00}} + w^k\partial_k\underline{h_{00}}\big) + 3\big(2\underline{h_{00}} - w^k\partial_k\underline{h_{00}}\big)\right](w^0)^2\ .$$

On the light cone this gives

$$\overline{\underline{h_{ij}}}\Theta^i = -\overline{\underline{h_{0j}}}\ , \tag{2.43}$$

$$\overline{\underline{h_{0j}}}\Theta^i = -\overline{\underline{h_{00}}}\ , \tag{2.44}$$

$$\frac{1}{r^2}\overline{\underline{h_{ij}}}\Theta^i\Theta^j = \frac{1}{r^2}\overline{\underline{h_{00}}}\ , \tag{2.45}$$

$$\frac{1}{r}\Theta^k\Theta^i\Theta^j\overline{\partial_k\underline{h_{ij}}} = -\frac{2}{r^2}\overline{\underline{h_{00}}} + \Theta^k\overline{\partial_k\underline{h_{00}}}\ , \tag{2.46}$$

$$\Theta^\ell\Theta^k\Theta^i\Theta^j\overline{\partial_\ell\partial_k\underline{h_{ij}}} = \frac{1}{r^2}\overline{w^\ell\partial_\ell\big(-2\underline{h_{00}} + w^k\partial_k\underline{h_{00}}\big) + 3\big(2\underline{h_{00}} - w^k\partial_k\underline{h_{00}}\big)}\ . \tag{2.47}$$

Since all the right-hand sides are mildly singular, from (2.38) we conclude that

$$\begin{aligned}\mathring{\Delta}(\mathring{\Delta}+2)\alpha &= r^{-2}\mathring{s}^{AC}\mathring{s}^{BD}\chi_{CD||AB} = r^2\eta^{AC}\eta^{BD}\chi_{CD||AB} = r^2\chi^{AB}{}_{||AB} \\ &= r^2 \times \text{m.s.}\ ;\end{aligned} \tag{2.48}$$

equivalently,

$$\mathring{\Delta}(\mathring{\Delta}+2)\alpha \text{ is } C_O\text{–smooth.}$$

Up to an element of the kernel of $\mathring{\Delta}(\mathring{\Delta}+2)$, which is irrelevant as it does not contribute to (2.27), we find that α is C_O–smooth: Indeed, if we let Π denote the projector, at fixed r, on the space orthogonal to $\ell=0$ and $\ell=1$ spherical harmonics, we have

Proposition 2.5 *Let $k \in \mathbf{N}\cup\{\infty\}\cup\{\omega\}$, and let $\mathring{\Delta}(\mathring{\Delta}+2)\alpha \in C^k(C_O)$. Then*

$$\Pi\alpha \in C^k(C_O)\ .$$

PROOF: Assume, first, that $k<\infty$. Let

$$\sum_{p=2}^{k}\big(f_{i_1\cdots i_p}\Theta^{i_1}\cdots\Theta^{i_p} + {f'}_{i_1\cdots i_{p-1}}\Theta^{i_1}\cdots\Theta^{i_{p-1}}\big)r^p + o_k(r^k) \qquad (2.49)$$

be the Taylor series of $\mathring{\Delta}(\mathring{\Delta}+2)\alpha$, as guaranteed by Lemma A.1. (The fact that the series starts at $p=2$ will be justified shortly.) Decomposing the coefficients $f_{i_1\ldots i_p}$ and ${f'}_{i_1\ldots i_{p-1}}$ into trace terms and trace-free parts, and rearranging the result, we can without loss of generality assume that the $f_{i_1\ldots i_p}$'s and ${f'}_{i_1\ldots i_{p-1}}$'s are traceless. It then follows from [5, pp. 201-202] that the finite sums

$$\sum_{p\ \text{fixed}} f_{i_1\ldots i_p}\Theta^{i_1}\cdots\Theta^{i_p} \text{ and } \sum_{p\ \text{fixed}} {f'}_{i_1\ldots i_{p-1}}\Theta^{i_1}\cdots\Theta^{i_{p-1}} \qquad (2.50)$$

are linear combinations of $\ell=p$, respectively $\ell=p-1$, spherical harmonics. (This explains why the sum in (2.49) starts with $p=2$, as the image of $\mathring{\Delta}(\mathring{\Delta}+2)$ is orthogonal to $\ell=0$ and $\ell=1$ spherical harmonics.) Set

$$\begin{aligned}\varphi := \alpha - \sum_{p=2}^{k}&\frac{1}{p(p+1)(p+2)}\\ &\times\Big[\frac{1}{(p+3)}f_{i_1\ldots i_p}\Theta^{i_1}\cdots\Theta^{i_p} + \frac{1}{(p-1)}{f'}_{i_1\ldots i_{p-1}}\Theta^{i_1}\cdots\Theta^{i_{p-1}}\Big]r^p\ .\end{aligned}$$

Then

$$\mathring{\Delta}(\mathring{\Delta}+2)\varphi = o_k(r^k)\ .$$

Standard elliptic estimates imply that

$$\forall\ 0 \le i \le k \quad \|\partial_r^i \Pi\varphi\|_{H^{k-i}(S^2)} = o(r^{k-i}) ,$$

and our claim easily follows.

If $k=\omega$, convergence for small $|w^0|+|\vec{w}|$ of the series

$$\sum_{p=2} \frac{1}{p(p+1)(p+2)} \big[\frac{1}{(p+3)} f_{i_1\ldots i_p} w^{i_1}\cdots w^{i_p} + w^0 \frac{1}{(p-1)} {f'}_{i_1\ldots i_{p-1}} w^{i_1}\cdots w^{i_{p-1}} \big] \tag{2.51}$$

follows immediately from that of

$$\sum_{p=2} \big(f_{i_1\ldots i_p} \Theta^{i_1}\cdots\Theta^{i_p} + {f'}_{i_1\ldots i_{p-1}} \Theta^{i_1}\cdots\Theta^{i_{p-1}} \big) r^p .$$

If $k=\infty$ we let $\tilde\alpha$ denote the Borel sum, as in Appendix D, associated with (2.51). Then

$$\forall k \qquad \mathring{\Delta}(\mathring{\Delta}+2)(\alpha-\tilde\alpha) = o_k(r^k) ,$$

and one concludes as before. □

Returning to our main argument, note that it follows from (2.36) and (2.45) that

$$\gamma = \frac{1}{2}\overline{\eta^{\mu\nu}h_{\mu\nu}} , \tag{2.52}$$

which shows that γ is C_O–smooth.

We pass now to the term

$$\mathring{\nabla}_\rho \left(\eta^{\sigma\rho} T_\alpha w_\beta \epsilon^{\alpha\beta\gamma\delta} \mathring{\nabla}_\gamma h_{\delta\sigma} \right) .$$

Let $\epsilon_{\mu\nu\rho\sigma}$ be the unique anti-symmetric tensor such that

$$\underline{\epsilon_{0123}} = 1 , \quad \text{we set } \epsilon^{AB} = \frac{w^i}{r} \underline{\epsilon^{0ijk}} \frac{\partial x^A}{\partial w^j} \frac{\partial x^B}{\partial w^k} .$$

Here, and in what follows, we use the summation convention on any repeated indices, regardless of their positions. We have

$$T_\alpha w_\beta \epsilon^{\alpha\beta\gamma\delta} \mathring{\nabla}_\gamma h_{\delta\sigma} = -w^i \epsilon^{0ijk} \mathring{\nabla}_j h_{k\sigma} = r\epsilon^{AB} \mathring{\nabla}_B h_{A\sigma} = r\epsilon^{AB} h_{A\sigma;B} ,$$

$$h_{AC;B} = h_{AC||B} + \frac{1}{r}\eta_{AB}(h_{uC} + h_{rC}) + \frac{1}{r}\eta_{BC}(h_{uA} + h_{rA}) \, ,$$
$$h_{Aa;B} = h_{Aa||B} + \frac{1}{r}\eta_{AB}(h_{ua} + h_{ra}) - \delta^r_a \frac{1}{r} h_{AB}$$
$$\epsilon^{AB} h_{AC;B} = \epsilon^{AB} h_{AC||B} + \frac{1}{r}\epsilon^{A}{}_{C}(h_{uA} + h_{rA}) \, ,$$
$$\epsilon^{AB} h_{Aa;B} = \epsilon^{AB} h_{Aa||B} \, ,$$

and finally

$$\begin{aligned}
&\mathring{\nabla}_\rho \left(\eta^{\sigma\rho} T_\alpha w_\beta \epsilon^{\alpha\beta\gamma\delta} \mathring{\nabla}_\gamma h_{\delta\sigma}\right) = \left(r\epsilon^{AB} h_{A\sigma;B}\right)^{;\sigma} \\
&= \frac{1}{r^2}\left(r^3 \eta^{ab}\epsilon^{AB} h_{Ab;B}\right)_{,a} + \left(r\eta^{CD}\epsilon^{AB} h_{AC;B}\right)_{||D} \\
&= r\epsilon^{AB}\chi_A{}^C{}_{||BC} + r\epsilon^{AB}\partial_u \underbrace{h^u{}_{A||B}}_{h_{rA||B}} + \frac{1}{r^3}\partial_r\left(r^4\epsilon^{AB}\underbrace{h^r{}_{A||B}}_{h_{uA||B}+h_{rA||B}}\right) \, ,
\end{aligned} \tag{2.53}$$

where, as before, χ_{AB} is the traceless part of h_{AB}.

The left-hand side of the last equation is a smooth function on space-time. Next,

$$r\epsilon^{AB} h_{Ar;B} = \underline{T_\alpha w_\beta \epsilon^{\alpha\beta\gamma\delta}\mathring{\nabla}_\gamma h_{\delta\sigma}}\, dw^\sigma(\partial_r) = \underline{T_\alpha w_\beta \epsilon^{\alpha\beta\gamma\delta}\mathring{\nabla}_\gamma h_{\delta 0}} + \frac{w^i}{r}\underline{T_\alpha w_\beta \epsilon^{\alpha\beta\gamma\delta}\mathring{\nabla}_\gamma h_{\delta i}} \, ,$$

where the right-hand side is the sum of a smooth function and of a smooth function divided by r. Hence so is its $\partial_u = -\partial_{w^0}$–derivative, which is the second term in the last line of (2.53). We note the identity,

$$r\epsilon^{AB} h_{Au;B} = -\underline{T_\alpha w_\beta \epsilon^{\alpha\beta\gamma\delta}\mathring{\nabla}_\gamma h_{\delta 0}} \, ,$$

where the right-hand side is a smooth function on space-time. We conclude that

$$\epsilon^{AB}\chi_A{}^C{}_{||BC} \text{ is mildly singular.} \tag{2.54}$$

This implies that

$$\begin{aligned}
\mathring{\Delta}(\mathring{\Delta}+2)\beta &= \overline{r^{-2}\mathring{\epsilon}^{AB}\mathring{s}^{CD}\chi_{AD||BC}} \\
&= \overline{r^2\epsilon^{AB}\eta^{CD}\chi_{AD||BC}} \\
&= \overline{r^2\epsilon^{AB}\chi_A{}^C{}_{||BC}} \\
&= r^2 \times \text{m.s.}
\end{aligned} \tag{2.55}$$

Up to an element of the kernel of $\mathring{\Delta}(\mathring{\Delta}+2)$, which is irrelevant as it does not contribute to (2.27), we find that β is C_O–smooth. We have therefore proved necessity in Theorem 1.1.

We wish to show, now, that the conditions of our statement are sufficient: C_O–smooth functions α, β, and γ lead to smooth metrics in normal coordinates. For this, it is convenient to view tensors on S^2 as tensors on $\mathbf{R}^3$ which are orthogonal to y^i in all indices. For example, the metric $\mathring{s} = \mathring{s}_{AB}dx^A dx^B$ is identified with r^{-2} times the projector

$$P_{ij} = \delta_{ij} - \frac{w^i w^j}{r^2} \ .$$

Indeed,

$$\mathring{s}_{AB}dx^A dx^B = \mathring{s}_{AB}\frac{\partial x^A}{\partial w^i}\frac{\partial x^B}{\partial w^j}dw^i dw^j = r^{-2}\left(\delta_{ij} - \frac{w^i w^j}{r^2}\right)dw^i dw^j \ .$$

So, if Y_i or S_{ij} are tensors satisfying $Y_i w^i = 0 = S_{ij}w^j = S_{ij}w^i$, we have the formulae

$$\mathcal{D}_i Y_j = P_i{}^k P_j{}^\ell \partial_k Y_\ell \ , \quad \mathcal{D}_i S_{jm} = P_i{}^k P_j{}^\ell P_m{}^n \partial_k S_{\ell n} \ , \quad \mathcal{D}_i S^i{}_m = P_\ell{}^k \partial_k S_{\ell n} \ .$$

In this formalism we have $\mathcal{D}_i f = P_i{}^j \partial_j f$, and

$$\mathcal{D}_i\mathcal{D}_j f \ = \ P_i{}^k P_j{}^\ell \partial_k (P_\ell{}^m \partial_m f) = P_i{}^k P_j{}^\ell \partial_k \partial_\ell f - \frac{1}{r}P_{ij}\Theta^m \partial_m f \ .$$

Hence

$$P^{ij}\mathcal{D}_i\mathcal{D}_j f = P^{ij}\partial_i\partial_j f - \frac{2}{r}\Theta^m \partial_m f \ . \tag{2.56}$$

Let us write

$$\alpha = \check{\alpha} + r\hat{\alpha} \ ,$$

where $\check{\alpha}$ and $\hat{\alpha}$ are smooth functions of $\vec{w}$. We note that

$$\mathcal{D}_i\mathcal{D}_j\alpha = \mathcal{D}_i\mathcal{D}_j\check{\alpha} + r\mathcal{D}_i\mathcal{D}_j\hat{\alpha} \ .$$

Equation (2.56) with f replaced by $\check{\alpha}$ gives

$$\begin{aligned} & r^2\big(2\mathcal{D}_i\mathcal{D}_j\check{\alpha} - P_{ij}P^{k\ell}\mathcal{D}_k\mathcal{D}_\ell\check{\alpha}\big) \\ & \quad = \ r^2\big(2P_i{}^k P_j{}^\ell \partial_k\partial_\ell\check{\alpha} - P_{ij}P^{k\ell}\partial_k\partial_\ell\check{\alpha}\big) \\ & \quad = \ \Theta^i\Theta^j w^k w^\ell \partial_k\partial_\ell\check{\alpha} - 2w^i w^\ell \partial_j\partial_\ell\check{\alpha} - 2w^j w^\ell \partial_i\partial_\ell\check{\alpha} \\ & \qquad -(r^2\delta_i^j - w^i w^j)\partial_\ell\partial_\ell\check{\alpha} \ . \end{aligned} \tag{2.57}$$

An identical calculation applies to $\hat{\alpha}$. We conclude that the tensor field (1.1) contains $\Theta\otimes\Theta$ terms of the form as in (2.25), with

$$A_{ij} = \partial_i\partial_j\check{\alpha} + r\partial_i\partial_j\hat{\alpha}\ . \tag{2.58}$$

Next, we write

$$\beta = \check{\beta} + r\hat{\beta}\ ,$$

where $\check{\beta}$ and $\hat{\beta}$ are smooth functions of $\vec{w}$. The contribution of $\check{\beta}$ to the tensor field (1.1) can be rewritten as

$$\begin{aligned} & rw^\ell\left(\epsilon_{k\ell i}\partial_{w^j}\partial_{w^k}\check{\beta} + \epsilon_{k\ell j}\partial_{w^i}\partial_{w^k}\check{\beta}\right) \\ & + w^\ell w^m\left(\Theta^i\epsilon_{jk\ell} + \Theta^j\epsilon_{ik\ell}\right)\partial_{w^k}\partial_{w^m}\check{\beta}\ , \end{aligned}$$

with a similar formula for $\hat{\beta}$. The resulting Θ terms are of the right-form

$$w^m w^\ell\big(\underline{A_{\ell im}}\Theta^j + \underline{A_{\ell jm}}\Theta^i\big)$$

as in (2.25) if we set

$$A_{\ell jm} = \epsilon_{mk\ell}(\partial_{w^j}\partial_{w^k}\check{\beta} + r\partial_{w^j}\partial_{w^k}\hat{\beta}) - \epsilon_{jk\ell}(\partial_{w^m}\partial_{w^k}\check{\beta} + r\partial_{w^m}\partial_{w^k}\hat{\beta})\ . \tag{2.59}$$

To summarize: let Ω_{0i0j} be a smooth extension of A_{ij} as given by (C.9), and let Ω_{0ijk} be a smooth extension of A_{ijk} as given by (2.59), if we set $\Omega_{ijkl}=0$, then the restrictions to the light cone of the ij components of the tensor field

$$(\underline{\eta_{\mu\nu}} + \underline{\Omega_{\mu\alpha\rho\beta}}w^\alpha w^\beta)dw^\mu dw^\nu$$

reproduce the non-manifestly C_O–smooth terms in

$$r^2\big[(1+\gamma)\mathring{s}_{AB} + 2\alpha_{||AB} - \mathring{s}_{AB}\mathring{s}^{CD}\alpha_{||CD} + \mathring{\epsilon}_A{}^C\beta_{||CB} + \mathring{\epsilon}_B{}^C\beta_{||CA}\big]\frac{\partial x^A}{\partial w^i}\frac{\partial x^B}{\partial w^j}\ .$$

So the difference is a C_O–smooth tensor field, say $f_{ij} = \hat{f}_{ij} + r\check{f}_{ij}$, with $\hat{f}_{ij}$ and $\check{f}_{ij}$ smooth tensors on $\mathbf{R}^3$, that satisfies

$$f_{ij}w^j = 0\ . \tag{2.60}$$

Now, it is not directly apparent that we have the desired formula, as in Proposition 2.2,

$$\underline{f_{ij}} = \underline{A_{ikj\ell}}\,w^k w^\ell \tag{2.61}$$

for some tensor field A_{ijkl} with the right symmetries, because f_{ij} is not differentiable. However, one can proceed as follows: Let $\hat{f}_{ijk_1\ldots k_\ell}$ be the Taylor expansion coefficients of $\hat{f}$,

$$\forall m \qquad \hat{f}_{ij}(\vec{w}) = \sum_{0\le \ell\le m} \hat{f}_{ijk_1\ldots k_\ell} w^{k_1}\cdots w^{k_\ell} + o_m(r^m) \ ,$$

similarly for $\check{f}_{ijk_1\ldots k_\ell}$. Then the coefficients in the Taylor expansion of $f_{ij}w^i$ have to vanish at every power of r, which implies that for all $\ell\in\mathbf{N}$ we have

$$\sum_{\text{fixed }\ell} \big(\hat{f}_{ijk_1\ldots k_\ell}\Theta^{k_1}\cdots\Theta^{k_\ell} + \check{f}_{ijk_1\ldots k_{\ell-1}}\Theta^{k_1}\cdots\Theta^{k_{\ell-1}}\big)\Theta^i = 0 \ .$$

Equivalently,

$$\sum_{\text{fixed }\ell} \big(\hat{f}_{ijk_1\ldots k_\ell}w^{k_1}\cdots w^{k_\ell} + r\check{f}_{ijk_1\ldots k_{\ell-1}}w^{k_1}\cdots w^{k_{\ell-1}}\big)w^i = 0 \ .$$

Comparing this equation with the equation where w^k is replaced by $-w^k$ we easily conclude that

$$\hat{f}_{i(jk_1\ldots k_\ell)} = 0 = \check{f}_{i(jk_1\ldots k_\ell)} \ .$$

Let $\widetilde{\check{f}}_{ij}$ be obtained by Borel summation of the Taylor series of $\check{f}_{ij}$, as in Appendix D. Then each partial sum $(\widetilde{\check{f}}_{ij})_p$ as defined in (D.1) has vanishing contraction with w^i, and so $\widetilde{\check{f}}_{ij}w^i = 0$ as well by passing to the limit. Since $\check{f}_{ij}$ and $\widetilde{\check{f}}_{ij}$ have the same Taylor coefficients it holds that

$$\forall\ m \qquad \check{f}_{ij} - \widetilde{\check{f}}_{ij} = o_m(r^m) \ ,$$

where we write $\psi = o_m(r^m)$ if ψ is m–times differentiable with $\lim_{r\to 0}\partial_{k_1}\cdots\partial_{k_\ell}\psi = 0$ for $0\le\ell\le m$. This implies that $r(\check{f}_{ij} - \widetilde{\check{f}}_{ij})$ is smooth. Hence

$$\hat{f}_{ij} + r(\check{f}_{ij} - \widetilde{\check{f}}_{ij})$$

is a smooth tensor field satisfying

$$\big[\hat{f}_{ij} + r(\check{f}_{ij} - \widetilde{\check{f}}_{ij})\big]w^i = 0 \ .$$

By Proposition 2.2 we can write

$$\hat{f}_{ij} + r(\check{f}_{ij} - \widetilde{\check{f}}_{ij}) = \hat{A}_{ikj\ell}w^k w^\ell \ , \quad \widetilde{\check{f}}_{ij} = \check{A}_{ikj\ell}w^k w^\ell \ .$$

This shows that

$$f_{ij} = (\underbrace{\hat{A}_{ikj\ell} + r\check{A}_{ikj\ell}}_{=:A_{ikj\ell}})w^k w^\ell \ ,$$

as desired.

One concludes using Proposition 2.3. $\square$

3 Other adapted coordinate systems

So far we have concentrated on normal coordinates, as these are naturally singled out by the geometry. However, other (local) coordinate systems y^μ in which C_O takes the standard form $\{y^0 = |\vec{y}|\}$ exist, and can be useful for some purposes. The simplest possibility is provided by coordinate systems of the form

$$y^\mu = w^\mu + \underline{\eta_{\alpha\beta}} w^\alpha w^\beta \chi^\mu \ , \tag{3.1}$$

for some smooth functions χ^μ. It is likely that all coordinate systems for which $C_O = \{y^0 = |\vec{y}|\}$ are related to the normal ones in this way, but we are not aware of a proof of this except in the analytic case in dimension $3+1$.

For sufficiently small $|w^0| + |\vec{w}|$ the inverse transformation to (2.51) takes a similar form

$$w^\mu = y^\mu + \underline{\eta_{\alpha\beta}} y^\alpha y^\beta \psi^\mu \ , \tag{3.2}$$

for some smooth functions ψ^μ.

To avoid ambiguities, let us write

$$g = g_{y^\mu y^\nu} dy^\mu dy^\nu = g_{w^\mu w^\nu} dw^\mu dw^\nu \equiv \underline{g_{\mu\nu}} dw^\mu dw^\nu \ ;$$

one finds

$$\overline{g_{y^\mu y^\nu}} = \overline{g_{w^\mu w^\nu}} + 2\overline{g_{w^\alpha w^\nu}\chi^\alpha} y_\mu + 2\overline{g_{w^\alpha w^\mu}\chi^\alpha} y_\nu + 4\overline{g_{w^\alpha w^\beta}\chi^\alpha\chi^\beta} y_\mu y_\nu \ ,$$

where $y_\alpha = \eta_{y^\alpha y^\beta} y^\beta$, with $\eta_{y^\mu y^\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$. Clearly $\{\eta_{y^\mu y^\nu} y^\mu y^\nu = 0\}$ remains a null hypersurface on geometric grounds; a useful consistency check in subsequent calculations is to note that the last equation implies

$$\overline{g_{y^\mu y^\nu}} y^\nu = \overline{g_{w^\mu w^\nu}} y^\nu = \eta_{y^\mu y^\nu} y^\nu \ . \tag{3.3}$$

To avoid a proliferation of notation, we will again use the symbols x^α to denote coordinates defined as

$$y^0 = x^1 - x^0, \quad y^i = x^1 \Theta^i(x^A) \quad \text{with, as before,} \quad \sum_{i=1}^n \left[\Theta^i(x^A)\right]^2 = 1 \ . \tag{3.4}$$

It follows from (2.20)-(2.23) that the new $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ takes on C_O the form

$$\begin{aligned}
\overline{h_{00}} &= \underline{\overline{\Omega_{0i0j}}}y^i y^j - 4r\underline{\overline{g_{\mu 0}\chi^\mu}} + 4r^2\underline{\overline{g_{\mu\nu}\chi^\mu\chi^\nu}} \;, & (3.5)\\
\overline{h_{01}} &= 2\big(r\underline{\overline{g_{\mu 0}\chi^\mu}} - \underline{\overline{g_{\mu i}\chi^\mu y^i}}\big) \;, & (3.6)\\
\overline{h_{1A}} &= \overline{h_{11}} = 0 \;, & (3.7)\\
\overline{h_{0A}} &= \Big[r\big(\underline{\overline{\Omega_{0j0i}}}r - \underline{\overline{\Omega_{0jik}}}y^k\big)y^j + 2r^2\underline{\overline{g_{\mu i}\chi^\mu}}\Big]\frac{\partial\Theta^i}{\partial x^A} \;, & (3.8)\\
\overline{h_{AB}} &= r^2\Big[\underline{\overline{\Omega_{i0j0}}}r^2 \\
&\quad +\big(-\underline{\overline{\Omega_{0ijk}}}r - \underline{\overline{\Omega_{0jik}}}r + \underline{\overline{\Omega_{ikj\ell}}}y^\ell\big)y^k\Big]\frac{\partial\Theta^i}{\partial x^A}\frac{\partial\Theta^j}{\partial x^B} \;. & (3.9)
\end{aligned}$$

A Extending functions

Lemma A.1 *A function φ defined on a light cone C_O is the trace $\overline{f}$ on C_O of a C^k spacetime function f if and only if φ admits an expansion, for small r, of the form*

$$\varphi = \sum_{p=0}^{k} f_p r^p + o_k(r^k) \;, \tag{A.1}$$

with

$$f_p \equiv f_{i_1\ldots i_p}\Theta^{i_1}\cdots\Theta^{i_p} + f'_{i_1\ldots i_{p-1}}\Theta^{i_1}\cdots\Theta^{i_{p-1}} \;, \tag{A.2}$$

where $f_{i_1\ldots i_p}$ and $f'_{i_1\ldots i_{p-1}}$ are numbers.

The claim remains true with $k=\infty$ if (A.1) holds for all k.

Proof: The result is trivial away from the origin, so it suffices to consider functions defined near the tip of the light cone.

Suppose, first, that $k<\infty$. To see the necessity, let f be a function which is C^k in a neighbourhood of the origin in $\mathbf{R}^{n+1}$. For any multi-index $\beta = (\beta_1,\ldots,\beta_j)\in(\mathbf{N}^{n+1})^j$, $\beta_i\in\{0,1,\ldots,n\}$, with length $1\le|\beta| := j\le k$ set

$$f_\beta := \frac{\partial}{\partial y^{\beta_1}}\cdots\frac{\partial}{\partial y^{\beta_j}} f \;.$$

Then f_β is $C^{k-|\beta|}$ in a neighbourhood of the origin, and thus admits a Taylor expansion

$$f_\beta = \underbrace{\sum_{p=0}^{k-|\beta|} h_{\beta;\alpha_1\cdots\alpha_p} y^{\alpha_1}\cdots y^{\alpha_p}}_{=:h_\beta} + \underbrace{g_\beta}_{o(|y|^{k-|\beta|})} \ , \qquad \text{(A.3)}$$

for some coefficients $h_{\beta;\alpha_1\cdots\alpha_p} \in \mathbf{R}$. Since $f_\beta \in C^{k-|\beta|}$ and $h_\beta \in C^\infty$ we have $g_\beta = f_\beta - h_\beta \in C^{k-|\beta|}$. Similarly

$$f = \underbrace{\sum_{p=0}^{k} f_{\alpha_1\cdots\alpha_p} y^{\alpha_1}\cdots y^{\alpha_p}}_{=:h} + \underbrace{g}_{o(|y|^k)} \ , \qquad \text{(A.4)}$$

with $f_{\alpha_1\cdots\alpha_p} \in \mathbf{R}$, $h \in C^\infty$ and $g \in C^k$. The usual formula for the coefficients of a Taylor expansion implies that

$$h_\beta = \frac{\partial}{\partial y^{\beta_1}}\cdots\frac{\partial}{\partial y^{\beta_j}} h \ .$$

Hence

$$\begin{aligned} \frac{\partial}{\partial y^{\beta_1}}\cdots\frac{\partial}{\partial y^{\beta_j}} g &= \frac{\partial}{\partial y^{\beta_1}}\cdots\frac{\partial}{\partial y^{\beta_j}} (f-h) \\ &= f_\beta - h_\beta = g_\beta = o(|y|^{k-j}) \ , \qquad \text{(A.5)} \end{aligned}$$

and so $g = o_k(|y|^k)$. Now, $\overline{f} = \overline{h} + \overline{g}$, and it should be clear that $\overline{h}$ is of the form (A.2). The estimate $\overline{g} = o_k(r^k)$ is then straightforward from $g = o_k(|y|^k)$, using

$$\frac{\partial}{\partial y^{i_1}}\cdots\frac{\partial}{\partial y^{i_j}}\overline{g} = \overline{\left(\frac{y^{i_1}}{r}\frac{\partial}{\partial y^0} + \frac{\partial}{\partial y^{i_1}}\right)\cdots\left(\frac{y^{i_j}}{r}\frac{\partial}{\partial y^0} + \frac{\partial}{\partial y^{i_j}}\right) g} \ .$$

Conversely, let $\varphi = \psi + \chi$ be defined on a neighbourhood of O on C_O, where

$$\begin{aligned} \psi &= \sum_{p=0}^{k} (f_{i_1\ldots i_p}\Theta^{i_1}\cdots\Theta^{i_p} + f'_{i_1\ldots i_{p-1}}\Theta^{i_1}\cdots\Theta^{i_{p-1}})r^p \\ &= \sum_{p=0}^{k} (f_{i_1\ldots i_p} y^{i_1}\cdots y^{i_p} + r f'_{i_1\ldots i_{p-1}} y^{i_1}\cdots y^{i_{p-1}}) \ , \end{aligned}$$

and where $\chi = o_k(r^k)$. Set $t = y^0$, $\vec{y} = (y^1, \dots, y^n)$, and

$$f(t, \vec{y}) = \sum_{p=0}^{k} (f_{i_1 \dots i_p} y^{i_1} \cdots y^{i_p} + t {f'}_{i_1 \dots i_{p-1}} y^{i_1} \cdots y^{i_{p-1}}) + \chi \ .$$

Then $\overline{f} = \varphi$. The function $\chi(\vec{y})$, viewed as a function of $(t, \vec{y})$, is trivially $o_k(|y|^k)$, and the proof is completed for finite k.

The case $k = \infty$ is obtained from the above by Borel summation, using Lemma D.1, Appendix D. $\Box$

B How to recognize that coordinates are normal

In this appendix we prove some simple necessary and sufficient conditions for a coordinate system to be normal:

Proposition B.1 (Thomas [10]) *Let $\{x^\mu\}$ be a local coordinate system defined on a star shaped domain containing the origin. The following conditions are equivalent:*

1. *For every $a^\mu \in \mathbf{R}^n$ the rays $s \to sa^\mu$ are geodesics;*
2. *${\Gamma^\mu}_{\alpha\beta}(x) x^\alpha x^\beta = 0$;*
3. *$\dfrac{\partial g_{\gamma\alpha}}{\partial x^\beta}(x) x^\alpha x^\beta = 0$;*
4. *$g_{\alpha\beta}(x) x^\beta = g_{\alpha\beta}(0) x^\beta$.*

Proof: 1. $\Leftrightarrow$ 2.: The rays $\gamma^\mu(s) = sa^\mu$ are geodesics if and only if

$$0 = \underbrace{\frac{d^2\gamma^\mu}{ds^2}}_{=0} + {\Gamma^\mu}_{\alpha\beta}(sa^\sigma) \frac{d\gamma^\alpha}{ds} \frac{d\gamma^\beta}{ds} = {\Gamma^\mu}_{\alpha\beta}(sa^\sigma) a^\alpha a^\beta \ ,$$

multiplying by s^2 and setting $x^\mu = sa^\mu$ the result follows.

3. $\Leftrightarrow$ 4.:

$$\begin{aligned} g_{\mu\alpha}(x^\sigma)x^\alpha = g_{\mu\alpha}(0)x^\alpha \quad &\Longleftrightarrow \quad g_{\mu\alpha}(sa^\sigma)a^\alpha = g_{\mu\alpha}(0)a^\alpha && \text{(B.1)} \\ &\Longleftrightarrow \quad \frac{d}{ds}\left(g_{\mu\alpha}(sa^\sigma)a^\alpha\right) = 0 && \text{(B.2)} \\ &\Longleftrightarrow \quad \frac{\partial g_{\mu\alpha}(x^\sigma)}{\partial x^\beta}x^\alpha x^\beta = 0\ . && \text{(B.3)} \end{aligned}$$

2. $\Rightarrow$ 4.: From the formula for the Christoffel symbols in terms of the metric we have

$$\Gamma^\mu{}_{\alpha\beta}(x)x^\alpha x^\beta = 0 \quad \Longleftrightarrow \quad \left(2\frac{\partial g_{\mu\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right)x^\alpha x^\beta = 0\ . \tag{B.4}$$

Multiplying by x^μ we obtain

$$\begin{aligned} \frac{\partial g_{\mu\alpha}(x^\sigma)}{\partial x^\beta}x^\alpha x^\beta x^\mu = 0 \quad &\Longleftrightarrow \quad \frac{\partial g_{\mu\alpha}(sa^\sigma)}{\partial x^\beta}a^\alpha a^\beta a^\mu = 0 && \text{(B.5)} \\ &\Longleftrightarrow \quad \frac{d}{ds}\left(g_{\mu\alpha}(sa^\sigma)a^\alpha a^\mu\right) = 0 && \text{(B.6)} \\ &\Longleftrightarrow \quad g_{\mu\alpha}(sa^\sigma)a^\alpha a^\mu = g_{\mu\alpha}(0)a^\alpha a^\mu && \text{(B.7)} \\ &\Longleftrightarrow \quad g_{\mu\alpha}(x^\sigma)x^\alpha x^\mu = g_{\mu\alpha}(0)x^\alpha x^\mu\ . && \text{(B.8)} \end{aligned}$$

Differentiating it follows that

$$\frac{\partial g_{\mu\alpha}(x^\sigma)}{\partial x^\gamma}x^\alpha x^\mu + 2g_{\gamma\alpha}(x^\sigma)x^\alpha = 2g_{\gamma\alpha}(0)x^\alpha\ .$$

Substituting this into the last term in (B.4) one obtains

$$\frac{\partial g_{\mu\alpha}}{\partial x^\beta}(x^\sigma)x^\alpha x^\beta + g_{\mu\alpha}(x^\sigma)x^\alpha - g_{\mu\alpha}(0)x^\alpha = 0\ . \tag{B.9}$$

This implies that

$$\frac{d}{ds}\left[g_{\mu\alpha}(sa^\mu)sa^\alpha - g_{\mu\alpha}(0)sa^\alpha\right] = 0\ ,$$

and the result follows by integration.

3.&4. $\Rightarrow$ 2.: Point 4 implies

$$g_{\alpha\beta}(x^\gamma)x^\alpha x^\beta = g_{\alpha\beta}(0)x^\alpha x^\beta\ .$$

Differentiating one obtains

$$\frac{\partial g_{\alpha\beta}(x^\gamma)}{\partial x^\mu} x^\alpha x^\beta + 2g_{\alpha\mu}(x^\gamma) x^\alpha = 2g_{\alpha\mu}(0) x^\alpha \ .$$

The last two terms are equal by point 4 so that

$$\frac{\partial g_{\alpha\beta}(x^\gamma)}{\partial x^\mu} x^\alpha x^\beta = 0 \ .$$

This shows that the last term in (B.4) vanishes, so does the next-to-last by point 3, and the proof is complete. $\square$

C Covector fields

The aim of this appendix is to present a simple equivalent of our parameterization of the metric for covector fields. This can be used for Cauchy problems on the light cone involving Maxwell fields.

We start by noting that every covector field ζ_μ on space-time can be written as

$$\zeta_\mu = \xi_\mu + \partial_\mu \lambda \ , \ \text{with } w^\mu \xi_\mu = 0 \ ,$$

for a smooth function λ. This is obtained by setting

$$\lambda(w^\mu) = w^\alpha \int_0^1 \zeta_\alpha(s w^\mu) ds \ .$$

By the arguments in Section 2.1 there exists a smooth anti-symmetric matrix $\Omega_{\mu\nu}$ such that

$$\underline{\xi_\mu} = \underline{\Omega_{\mu\nu}} w^\nu \ . \tag{C.1}$$

As in the main body of this paper, the restriction to the light cone $\{w^0 = |\vec{w}|\}$ of ξ_μ arises from a smooth vector field on $\mathbf{R}^4$ satisfying $w^\mu \underline{\xi_\mu} = 0$ if and only if the restrictions $\overline{\Omega_{\mu\nu}}$ are C_O–smooth.

An alternative parameterization of ξ is obtained by introducing

$$\xi_u = -\underline{\xi_0} \ , \quad \gamma = \Theta^i \overline{\underline{\xi_i}} \ , \quad \xi_A = \underline{\xi_i} w^i_{,A} \ , \quad \overline{\xi_A} = \alpha_{||A} + \epsilon_A{}^C \beta_{||C} \ , \tag{C.2}$$

and we have $\overline{\xi_u} = \gamma$ in view of the condition $\xi_\mu w^\mu = 0$. We then have:

Theorem C.1 *A field of the form* (C.2) *defined on* $(0,R)\times S^2$ *is the restriction to the light cone* $\{w^0 = |\vec{w}|\}$ *of a smooth vector field on* $\mathbf{R}^4$ *satisfying* $w^\mu \underline{\xi_\mu} = 0$ *if and only if*

$$\alpha = r\check{\alpha} + r^2\hat{\alpha},\ \beta = r\check{\beta} + r^2\hat{\beta},\ \textit{and}\ \gamma = w^i\partial_{w^i}\check{\alpha} + r\check{\gamma} + r^2\hat{\gamma},$$

where

$$\check{\alpha},\ \hat{\alpha},\ \check{\beta},\ \hat{\beta},\ \check{\gamma}\ \textit{and}\ \hat{\gamma}\ \textit{are smooth functions of}\ \vec{w},$$

except for the $\ell = 0$ *spherical-harmonics components of* α *and* β *which do not affect* ξ_μ.

PROOF: Necessity: it follows from the identities

$$r^2\overline{\xi^A{}_{||A}} = \mathring{\Delta}\alpha = \overline{r^2\partial_{w^k}\underline{\xi^k} - w^j w^i \partial_{w^i}\underline{\xi_j} - 2rw^i\underline{\xi_i}}\ , \tag{C.3}$$

$$r^2\overline{\epsilon^{AB}\xi_{A||B}} = \mathring{\Delta}\beta = \overline{rw_i\epsilon^{ijk}\partial_{w^k}\underline{\xi_j}}\ , \tag{C.4}$$

together with a straightforward generalization of Proposition 2.5 that α and $r^{-1}\beta$ are C_O–smooth if ξ_μ is smooth, except for their $\ell = 0$ components which are in the kernel of $\mathring{\Delta}$. However, the gauge condition $0 = \underline{\xi_\mu}w^\mu = w^0\underline{\xi_0} + w^i\underline{\xi_i}$ implies

$$w^j w^i \partial_j \underline{\xi_i} = -tw^j\partial_j\underline{\xi_0} + t\underline{\xi_0}\ ,$$

and we conclude that

$$\overline{\Theta^i\underline{\xi_i}} = -\overline{\underline{\xi_0}}\ , \tag{C.5}$$

$$\overline{w^j w^i\partial_j\underline{\xi_i}} = -rw^j\overline{\partial_j\underline{\xi_0}} + r\overline{\underline{\xi_0}}\ . \tag{C.6}$$

The C_O–smoothness of γ follows from (C.5), while that of α/r follows from (C.3) and (C.6).

We can write

$$\gamma = w^i\partial_{w^i}\check{\alpha} + \psi\ ,$$

and it remains to show that ψ/r is C_O–smooth. The inverse of (C.2) reads

$$\begin{aligned}\overline{\underline{\xi_k}} \ &= \ r\partial_{w^k}\check{\alpha} + w^i\epsilon_{ik}{}^l\partial_{w^l}\check{\beta} + r^2\partial_{w^k}\hat{\alpha} + rw^i\epsilon_{ik}{}^l\partial_{w^l}\hat{\beta} - w^i\partial_{w^i}\hat{\alpha}w^k \\ &\quad +\psi\Theta^k\ .\end{aligned} \tag{C.7}$$

Extending $\hat{\alpha}$, $\check{\alpha}$, etc., to $\mathbf{R}^4$ by requiring the extension to be time-independent, and using the same symbols for this extension, $\overline{\underline{\xi_k}}$ minus the first line of the

right-hand side of (C.7) is the restriction to the light cone of the smooth vector field

$$\begin{aligned}\underline{\xi_k} \quad - \quad & \big(t\partial_{w^k}\check{\alpha} + w^i\epsilon_{ik}{}^l\partial_{w^l}\check{\beta} \\ & + r^2\partial_{w^k}\hat{\alpha} + tw^i\epsilon_{ik}{}^l\partial_{w^l}\hat{\beta} - w^i\partial_{w^i}\hat{\alpha}\, w^k\big) \ . \qquad \text{(C.8)}\end{aligned}$$

Hence for every k the function

$$\psi(\vec{w})\Theta^k = \frac{\psi(\vec{w})}{r}w^k$$

extends to a smooth function on space-time. Choosing k to be one, by Proposition 2.1 we can write

$$\frac{\psi(\vec{w})}{r}w^1 = \check{\chi}(\vec{w}) + r\hat{\chi}(\vec{w}) \ , \qquad \text{(C.9)}$$

for some smooth functions $\check{\chi}$ and $\hat{\chi}$. For $r \neq 0$ this implies

$$[\check{\chi}(\vec{w}) + r\hat{\chi}(\vec{w})]\,\big|_{w^1=0} = 0 \ ;$$

by continuity this holds for all r. Smoothness of $\check{\chi}$ and $\hat{\gamma}$ implies existence of smooth functions $\check{\gamma}$ and $\hat{\gamma}$ such that

$$\check{\chi} = \check{\chi}|_{w^1=0} + \check{\gamma}w^1 \ , \quad \hat{\chi} = \hat{\chi}|_{w^1=0} + \hat{\gamma}w^1 \ ,$$

and (C.9) gives

$$\frac{\psi(\vec{w})}{r}w^1 = \big[\check{\gamma}(\vec{w}) + r\hat{\gamma}(\vec{w})\big]w^1 \ . \qquad \text{(C.10)}$$

For $w^1 \neq 0$ we conclude

$$\psi(\vec{w}) = r\check{\gamma}(\vec{w}) + r^2\hat{\gamma}(\vec{w}) \ , \qquad \text{(C.11)}$$

and continuity implies that this equation holds everywhere. We conclude that ψ/r is C_O–smooth, and the proof of necessity is complete.

Sufficiency should be clear from what has been said together with

$$\psi(\vec{w})\Theta^k = \overline{\big[\check{\gamma}(\vec{w}) + t\hat{\gamma}(\vec{w})\big]w^k} \ . \qquad \text{(C.12)}$$

$\square$

D Borel's summation

In the main body of the paper we will need the details of the following construction, which is a straightforward adaption of [6, Volume I, Theorem 1.2.6]:

Lemma D.1 *[Borel summation] For any sequence* $\{c_{i_1\ldots i_k}\}_{k\in\mathbf{N}} = \{c, c_i, c_{ij}, \ldots\}$ *there exists a smooth function* f *such that, for all* $k \in \mathbf{N}$,

$$f - \sum_{p=0}^{k} c_{i_1\ldots i_p} y^{i_1} \cdots y^{i_p} = o_k(r^k) \ .$$

Proof. Let $\phi \in C^\infty(\mathbf{R})$ be any function such that

$$\phi|_{[0,1/2]} = 1 \ , \quad \phi|_{[1,\infty)} = 0 \ .$$

Set $f_0 = c$, and for $p > 1$

$$f_p = \sum_{i_1,\ldots,i_p} \phi(M_p|y|) c_{i_1\ldots i_p} y^{i_1} \cdots y^{i_p} \ , \tag{D.1}$$

where the constant M_p is chosen large enough so that for all $p > 0$ and for all multi-indices α satisfying

$$0 \le |\alpha| \le p - 1 \text{ we have } |\partial^\alpha f_p| \le 2^{-p} \ .$$

Then for each α the series

$$\sum_{p=0}^{\infty} \partial^\alpha f_p$$

is absolutely convergent. By standard results (see, e.g., [6, Volume I, Theorem 1.1.5]), the function

$$f := \sum_{p=0}^{\infty} f_p$$

is smooth, and is easily seen to have the required properties. □

Acknowledgements: Supported in part by the Polish Ministry of Science and Higher Education grant Nr N N201 372736. PTC acknowledges many useful discussions with Y. Choquet-Bruhat and Jose-Maria Martin Garcia on problems closely related to the ones addressed here.

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